Fractional Langevin model of memory in financial time series

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(Received 13 April 2001; published 6 March 2002)

Financial time series are random with the absolute value of the price index fluctuations having an inverse power-law correlation. A dynamical model of this behavior is proposed using a fractional Langevin equation. The physical basis for this model is the divergence of the microscopic time scale to overlap with the macroscopic time scale: a condition that is not observed in classical statistical mechanics. This time-scale separation provides a mechanism for the market to adjust the volitility of the price index fluctuations.

DOI: 10.1103/PhysRevE.65.037106 PACS number(s): 89.75.Da, 05.45.Tp, 89.65.Gh

In classical statistical physics the separation of the microscopic and macroscopic time scales is manifest in the central limit theorem. The separation of time scales implies that the macroscopic dynamics can be described by the ordinary stochastic differential calculus, even if the microscopic dynamics are incompatible with the methods of ordinary calculus [1]. When such a separation of time scales exists, the Langevin equation adequately describes the dynamics of the physical phenomenon. A similar conclusion was reached by Bachelier [2] in 1900, in his development of the phase-space equation of motion for the probability density characterizing the price fluctuations in the French stock market. The interpretation of financial markets using random walk models was given by Cootners in [3], in which an English translation of Bachelier's original paper resides.

On the other hand, when this separation of time scales does not exist, the formalism of ordinary statistical physics is no longer adequate to describe the phenomenon, as discussed for example by Grigolini, Rocco, and West [1]. In particular, a lack of time-scale separation may induce a fractional, stochastic, differential equation on the macroscopic level [1,9] to replace the Langevin equation. This is demonstrably the situation in financial markets where the time scales for individual events are quite small, and the variability in sign of price indices produce a very short correlation time. However, the magnitude of the price index changes can have very long memory, in fact, correlation overlap with the longest time scales in the financial market.

The price index fluctuation data from financial markets are not Gaussian, contrary to the early work in Ref. [3], but rather manifest distributions with fat tails [4]. The fluctuations in the index of prices have been modeled by a product function $g(t)=s(t)\eta(t)$, where $s(t)$ represents the changes in the sign of the price index and $\eta(t)$ the change in the magnitude of the price index [5]. It is well established that $g(t)$ has an exponentially short correlation time due to the variability in sign, whereas $\eta(t)$ has an inverse power-law correlation function. Such inverse power-law behavior has been observed in many other complex phenomena, such as crack growth [6], earthquakes [7], turbulence [8], and for a review of a number of others, see, for example, Ref. [9].

Given the absence of a universally accepted theoretical model of the dynamics of financial markets physicists interested in understanding the workings of complex phenomena have turned to uncovering any regularities that might be ascertainable through empirical laws [10]. Among the first to observe such regularities in a financial market context was Mandelbrot, see Ref. [4] for a review of early work and subsequent analysis. More recently, Stanley and his coworkers at Boston University have systematically processed one of the largest financial time series of a tick to tick nature in the literature, determining for market price index fluctuations: the correlational properties [10]; the statistical properties [11]; the statistical properties of volume fluctuations [12]; and the statistical properties for individual companies [13]. The results of their investigations agree with those of other researchers and indicate that the statistical behavior of financial markets cannot be described by the dynamics of simple diffusive processes, as thought by early investigators of the statistics of financial markets, see e.g., Montroll and Badger [14], but requires a more subtle analysis involving fractal statistical processes.

When the dynamical environment, to which a system is coupled, is fractal, for example, a fractal stochastic process, the dynamics of the system cannot be represented by the solution of an ordinary differential equation. However, the solutions to fractional differential operator equations still yield mathematically well-defined quantities. Furthermore, the application of fractional equations of motion to physical and social phenomena can be usefully interpreted in terms of memory effects [15]. Further, with a number of papers supporting the position that the stock market is a fractal environment in time [16,4], we argue that it is justified to employ a model of the dynamics of such a market using the Riemann-Liouville fractional operators [17]. In particular, we know that for a physical process with memory, the Langevin equation is generalized to the form

$$\frac{dv(t)}{dt}+\int_0^t K(t-t')v(t')dt'=f(t), \tag{1}$$

where the memory kernel determines the influence of the process between points in time, $f(t)$ is the random force and the two are related by means of a generalized fluctuation-dissipation relation [18]. Herein we incorporate the memory

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through a phenomenological memory kernal given by a fractional derivative rather than through Eq. (1). This approach has been taken by West, Bologna, and Grigolini [19] in physical systems, so the fractional Langevin equation we suggest is the one proposed first by Glöckle and Nonnenmacher [20] in a rheology context

$$ {}_0D_t^{\alpha}[v(t)]-v_0\frac{t^{-\alpha}}{\Gamma(1-\alpha)}=-\lambda^{\alpha}v(t)+\xi(t). \qquad (2)$$

$1\geqslant\alpha>0$, where $\xi(t)$ is a stochastic process, whose statistics must be specified and the initial value for the process is given by v_0. The operator ${}_0D_t^{\alpha}[\cdot]$ is chosen to be a Riemann-Liouville fractional derivative, see, for example, [17,20].

The general solution to the fractional Langevin Eq. (2) is given by [19–21]

$$v(t)=v_0E_{\alpha}(-[\lambda t]^{\alpha})+\int_0^t x^{-\alpha-1}E_{\alpha,\alpha}(-\lambda^{\alpha}x^{\alpha})\xi(t-x)dx, \qquad (3)$$

where the Green's function for the solution $E_{\alpha,\alpha}(x)$ is the generalized Mittag-Leffler function given by the series

$$E_{\alpha,\beta}(x)\equiv\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+\beta)}, \quad \alpha>0, \quad \beta>0. \qquad (4)$$

In the case $\beta=1$, the series reduces to the Mittag-Leffler function. In addition when $\alpha=1$, the Mittag-Leffler function becomes an exponential, so that the solution to the fractional Langevin equation can become identical to that for an Ornstein-Uhlenbeck process [21], but only when the ξ fluctuations are given by a Wiener process. For financial markets the dynamical variable is often given by the return $u(t)=\ln[p(t+T)/p(T)]$, where $p(t)$ is the price of a given stock at time t. The quantity of interest in our model (2) is the magnitude of the price changes $v(t)\equiv|u(t)|$, which essentially coincides with $\eta(t)$ mentioned previously. With this definition of the dynamical variable we have for the initial condition $v_0=0$ in Eqs. (2) and (3), and the fractional Langevin equation for the financial market only represents the magnitude of the price index variations. In our dynamical model, we only impose the condition that the random driving force is δ correlated and we do not constrain the process with a particular choice of statistical distribution.

The traditional quantities calculated from the magnitude of the logarithm of the price time series are the autocorrelation function and the standard deviation of the time series, both of which are regarded as measures of the volatility, depending on the context [22]. We can calculate these quantities using the solution to the fractional Langevin Eq. (3). The autocorrelation function $C(\tau,t)=\langle v(t)v(t-\tau)\rangle/\langle v(t)^2\rangle$ is constructed using Eq. (3), where the brackets denote averages over the ξ fluctuations, which are assumed to be δ correlated in time, with a finite magnitude. We express the autocorrelation function as a function of the dimensionless time difference $z=\tau/t$ to extract the dependence

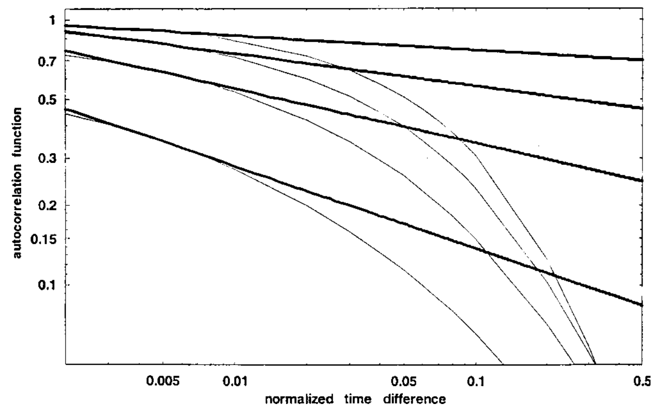

FIG. 1. Autocorrelation function (5) is plotted versus the dimensionless time interval z on log-log graph paper and a least-squares fit to the function for $0.001\leqslant z\leqslant 0.02$ with the phenomenological equation. Only the values $\alpha=0.9$, 0.8, 0.7, and 0.6 are shown.

of the autocorrelation function on the fractional derivative index. The analytic form of the autocorrelation function is determined to be [19]

$$C(\tau,t)=\frac{\displaystyle\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^{k+l-2}(\lambda l)^{k\alpha+l\alpha-1}(1-z)^{l\alpha}}{\Gamma(k\alpha)\Gamma(l\alpha+1)}F_{lk}(1-z)}{\displaystyle\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-|\lambda l|^{\alpha})^{k+l-2}}{\Gamma(k\alpha)\Gamma(l\alpha+1)}F_{lk}(1)}, \qquad (5)$$

where $F_{lk}(q)\equiv F(1;1-k\alpha;1+l\alpha:q)$ is the hypergeometric function. In Fig. 1 we observe the decrease in the autocorrelation function as a function of z. We fit this decrease with the phenomenological equation $C(z)=Az^{-B}$, over the range $0.001\leqslant z\leqslant 0.02$, where the empirical parameters A and B are functions of the fractional derivative index α. The parameters are obtained by a least-squares fit of the indicated phenomenological equation to the autocorrelation function (5). The values of the parameters for each value of α are fit in the indicated range yielding the coefficients $A=-1.18+2.03\alpha$ and $B=0.81-0.84\alpha$.

We can see from the figure that for early, but not too early, times each curve has a dominant inverse power-law form, but each with a different slope. Using the least-square fits, we write for a fixed-length time series the autocorrelation function in the interval $0.001t\leqslant\tau\leqslant 0.02t$ is

$$C(\tau)\propto\tau^{-0.81+0.84\alpha}, \qquad (6)$$

where, since $0<\alpha\leqslant 1$, we have an inverse power law in τ for most of the range of α. Here we can use the data analysis of Gopikrishnan *et al.* [10] for the correlation function of the absolute value of the price returns in their Fig. 3(b) or that of Liu *et al.* [11] in their Fig. 8(a). The value of the power-law index in the theoretical Eq. (6) is determined from Ref. [10] to be $0.81-0.84\alpha=0.30\pm0.08$, indicating a power-law index $\alpha=0.60\pm0.10$. Further, using a Tauberian theorem, we conclude that the high-frequency form of the spectrum is given by the inverse power law

$$S(\omega)\propto\omega^{-0.19-0.84\alpha} \tag{7}$$

as long as $\alpha<1$.

The autocorrelation function was devised as a quantitative measure of the linear dependence of the elements in Gaussian random processes. Thus, even when the autocorrelation function drops precipitously to zero, this does not mean that the price movements are statistically independent of one another, only that the price increments do not contain significant linear correlations, as has been well documented [23]. The autocorrelation function is somewhat more difficult to interpret when the statistics of the fluctuations are not Gaussian, but are of the inverse power-law form observed in financial markets. One possible interpretation is that the lack of exponential relaxation indicates that the memory process is not smooth. Rather, the inverse power law suggests a slipage or jerkiness to the system response, such as found in earthquakes, in the relaxation of stress in viscoelastic materials, in microcrack propagation, and in the cascade of energy in turbulent fluid flow [6]. This behavior is quite common in social phenomena [9], where there is a buildup of ''strain'' that often goes unnoticed until it is ''released,'' producing a significant change in the system dynamics. This trigger is often initiated by an innocuous event, thereby giving the event significance that is completely out of proportion to its true value. In this way the financial market responds strongly to the news that has been anticipated.

The integer value of the fractional derivative index $\alpha=1$ is a singular point. At $\alpha=1^{-}$ the phenomenological spectrum given by Eq. (7) is asymptotically given by $S(\omega)\propto 1/\omega$, that is, a spectrum corresponding to $1/f$ noise. As the index deviates from $1\alpha<1$, the more clustered the process becomes and the more the events in the distant past influence the present behavior of the financial market. Peters refers to this as pink noise [16]. Further, the greater the deviation from 1, the less reliable differential models of the market become. This is a consequence of the fact that the fractal dimension is a measure of the degree of irregularity of the time series. To understand this we note that the nearest neighbor correlation coefficient for a fractal process is $r=2^{3-2D}-1$ [9]. A completely correlated process has $r=1$ implying $D=1$, whereas an uncorrelated random process has $r=0$ implying $D=1.5$, the fractal dimension for Brownian motion [4]. Thus, as the fractal dimension increases from 1.0 to 1.5 the process becomes more and more irregular, manifesting less and less structure. The range of the fractal dimension is given by $1.19\geqslant D\geqslant 1.0$.

Note that a financial market is more than a collection of individual investors and, therefore, its collective behavior need not be the same as that of a ''reasonable'' average investor. The influence of the past events on the present events is found to be manifest in an intrinsic ''inertia'' in the market. A fractional derivation in the range $1>\alpha\geqslant 1/2$, such as found in the present fit to financial market data, implies that when the price fluctuation increases, the probability of increased volitility increases and when the price fluctuation decreases, the probability of increased volitility decreases. This mechanisms tends to slightly destablize the financial market, by adjusting the market strategy to amplify the influence of the change in the magnitude of the price index fluctuations and this behavior is reflected in the value of the fractal dimension $D=1.15$.

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